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MINIMAL IMPERFECT GRAPHS: A SIMPLE APPROACH

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We provide a very short proof of the following theorem of Lovász, and of its consequences: *A graph is perfect if and only if in every induced subgraph the number of vertices does not exceed the product of the stability and clique numbers of the subgraph.*

This proof is conceptually new: it does not use the "replication" operation, or any kind of polyhedral argument; the arguments resemble more the well-known ways of deducing structural properties of minimal imperfect graphs. However, the known proofs of these structural properties use Lovász's result, whereas the present work leads to a proof of Lovász's result itself, and actually to a slight sharpening.

Let G be a graph with vertex-set $V=V(G)$, and edge-set $E=E(G)$. We will use the notation $n=n(G):=|V(G)|$; $\omega=\omega(G)$ denotes the cardinality of a maximum clique of G; $\alpha=\alpha(G)$ denotes $\omega(\bar{G})$, that is the cardinality of a maximum stable set. If $k\in\mathbb{N}$, a k-clique or k-stable set will mean a clique or stable set of size k.

Let $\chi=\chi(G)$ be the chromatic number of G, that is the minimum number of stable sets partitioning V. A partitioning of G into k stable sets is called a k-*coloring*, and the elements of such a partition are called *color-classes*. A graph G is called *perfect* if $\chi(H)=\omega(H)$ for every induced subgraph H of G, otherwise it is called *imperfect*. Lovász [5] proved the following result, which was Berge's weak conjecture on perfect graphs [2]; it is also a basic theorem in polyhedral combinatorics, generalizing several minimax theorems (see for instance Fulkerson [4], Lovász [5], Lovász [7], Chvátal [3]).

Theorem 1. *A graph is perfect if and only if its complement is perfect.*

A graph G is called *minimal imperfect* if it is not perfect, but all its proper subgraphs are perfect. G is called an (α,ω)-*graph*, if $n=\alpha\omega+1$, $(\alpha,\omega\in\mathbb{N},\alpha\geq 2,\omega\geq 2)$, and $V(G)\setminus\{v\}$ can be partitioned both into ω-cliques and into α-stable sets,

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for every $v \in V(G)$. If G is an (α,ω)-graph, then $\chi = \omega + 1$, $\chi(G-v) = \omega = \omega(G-v)$, and $\bar{G}$ is also an (α,ω)-graph.

Lovász [6] proved a strengthening of Theorem 1: *a graph is perfect if and only if* $\alpha(H)\omega(H) \geq n(H)$ *for every induced subgraph.* This is trivially equivalent to the following (see for instance Lovász [7]):

Theorem 2. *If G is minimal imperfect, then it is an (α,ω)-graph.*

Lovász's proof of Theorem 2 is technically more involved than that of Theorem 1; on the other hand, it is just Theorem 2 which provides the only known coNP characterization of perfectness, and which is used by Padberg [8] to show further properties of minimal imperfect graphs.

Bland, Huang and Trotter [1] observed that the properties exhibited by Padberg hold for the larger class of (α,ω)-graphs.

These properties of minimal imperfect graphs are partial results in the proof we will be giving to Theorem 2. However, we leave it to the reader to gather the crops he finds interesting on the way, while we will be concentrating on proving Theorem 2 itself. Our proof uses widely the ideas of Padberg [8] and of Bland, Huang and Trotter [1]. Still, with no more work than these, it reproves Theorem 2 instead of using it.

Every minimal imperfect graph obviously satisfies (i) and (ii).

(i) For any stable set $S \subseteq V(G)$: $\omega(G-S) = \omega$.

(ii) There exists an α-stable set S_0 such that for all $s \in S_0$ $\chi(G-s) = \omega = \omega(G-s)$.

We will prove the following slight sharpening of Theorem 2.

Theorem 3. *If G satisfies (i) and (ii) then it is an (α,ω)-graph.*

Proof. Let G satisfy (i) and (ii), and $\mathcal{S} := \{S_0, S_1, \dots, S_{\alpha\omega}\}$, where S_0 is the stable set occuring in (ii); fixing an ω-coloration of each of the α graphs $G-s$ ($s \in S_0$), $S_1, \dots, S_{\alpha\omega}$ denote the stable sets occuring as a color-class in one of these colorations. Define $\mathcal{Q} := \{Q_0, Q_1, \dots, Q_{\alpha\omega}\}$, where Q_i is an ω-clique of $G - S_i$ (see (i) above). Repetitions are of course allowed in both $\mathcal{S}$ and $\mathcal{Q}$ (they are "multisets"), but it will actually turn out during the proof that there are none. Let I, J, A, B be the following matrices: I and J are $(\alpha\omega+1) \times (\alpha\omega+1)$ matrices, I is the identity matrix and J is the all 1 matrix; A and B are $(\alpha\omega+1) \times n$ matrices, where the i-th row of A is the characteristic vector of S_{i-1} (it has 1 for each element of S_{i-1}, 0 otherwise), whereas the i-th row of B is the characeristic vector of Q_{i-1} $(i = 1, \dots, \alpha\omega+1)$.

Claim 1. *Let $\{C_1, C_2, \dots, C_\omega\}$ be an ω-coloration of $G-v$ ($v \in V(G)$) and let K be an ω-clique of G. Then either:*

— *$v \notin K$, and $K \cap C_i \neq \emptyset$ $(i = 1, \dots, \omega)$*

— *$v \in K$ and $K \cap C_i = \emptyset$ for exactly one $i \in \{1, 2, \dots, \omega\}$.*

Indeed, $\omega = |K \cap \{v\}| + \sum_{i=1}^{\omega} |K \cap C_i|$, and each number of those sum is at most 1.

Claim 2. *Any ω-clique of G is disjoint from exactly one S_i $(i=0,\dots,\alpha\omega)$; $AB^T = J-I$.*

The first part of the Claim is immediate from Claim and the definition of the S_i's: either K is disjoint from S_0 and then meets all other S_i's or else there is exactly one S_i $(i=1,\dots,\alpha\omega)$ which is disjoint from K. Since $Q_i \cap S_i = \emptyset$ by definition, the second part follows.

Claim 3. *$n=\alpha\omega+1$; if $S\in\mathcal{S}$, then $|S|=\alpha$.*

Indeed, since $G-s$ $(s\in S_0)$ has a partition into ω stable sets, $n \le \alpha\omega+1$ is obvious. On the other hand, by Claim 2, A has full row rank, and $n \ge \alpha\omega+1$ follows. Thus, there is equality throughout, in particular all the color classes of $G-s$, $(s\in S_0)$ are of size α.

Claim 4. *$AB^T = B^TA = J-I$; $AJ=JA=\alpha J$; $BJ=JB=\omega J$.*

Two $n\times n$ matrices X and Y are said to *commute* if $XY=YX$.

It is obvious by the construction of $\mathcal{S}$ that every $v \in V$ is contained in α elements of $\mathcal{S}$: $JA=\alpha J$. $AJ=\alpha J$ is just the second part of Claim 3. So A and J commute.

Consequently A and $A^{-1}(J-I)=B^T$ also commute, so $B^TA=AB^T=J-I$ by Claim 2. Another consequence of $AJ=JA$ is that $B^T=A^{-1}(J-I)$ and J commute, and so $JB=BJ=\omega J$ from the construction.

Claim 5. *At least one of $S_i\cap S_j$ and $Q_i\cap Q_j$ is empty for all $i, j=0,1,\dots,\alpha\omega$, $i\neq j$.*

Indeed, if neither of them were empty, then the column of A corresponding to $s\in S_i\cap S_j$ and the column of B corresponding to $q\in Q_i\cap Q_j$ have both 1 in both their i-th and the j-th coordinate. Then entry (q,s) of B^TA is at least 2, contradicting Claim 4.

Let now $v\in V(G)$ be arbitrary. By Claim 4 there are α α-stable sets of $\mathcal{S}$ containing v, and from Claim 5 we get then that the ω-cliques of $\mathcal{Q}$ of the same indices form a partition of $V(G)\setminus\{v\}$. Similarly, by Claim 4, there are ω ω-cliques containing v, and then from Claim 5 we get a partition of $V(G)\setminus\{v\}$ into α-stable sets of $\mathcal{S}$. ∎

Note the asymmetry of the condition of Theorem 3 in contrast with the symmetry of its conclusion. Theorem 3 immediately implies Theorem 2 (actually Claim 3 already implies it), which in turn contains Theorem 1. The claims of the proof contain also all the properties of partitionable graphs shown by Padberg [8]. The most important of these, the fact that $\mathcal{Q}$ *is the set of all* ω-cliques of G follows directly from Claim 2: for any ω-clique Q, χ_Q is the unique solution of the equation $Ax=b$ where one coordinate of b is 0 and the others are 1. On the other hand, by Claim 2, one of the rows of B also satisfies this equation. In the same way $\mathcal{S}$ is the set of all α-stable sets.

Unlike in Theorem 2, the reverse implication in Theorem 3 is of course also true. (i) and (ii) are such that every minimal imperfect graph trivially satisfies them. On the other hand, they characterize (α,ω)-graphs.

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